DERIVATION of FRESENEL'S EQUATION from YOUNG'S PRINCIPLES

First case - incident light polarized in plane of incidence, displacement perpendicular to plane of incidence.
Let amplitude of displacement at the boundary interface be, f for the incident ray, g for the reflected ray, and h for the refracted ray.

Energy propogated across cross-section of incident, reflected and refracted beams are proportional to:

$$c_1\rho_1 f^2, \qquad c_1\rho_1 g^2, \qquad c_2\rho_2 h^2$$

where c_1, c_2, denote the velocities of light in the respective media, and ρ_1, ρ_2 the densities of the aether in the two media.

The cross-sections of the beams which meet the interface in unit area are: $\cos i, \quad \cos i, \quad \cos r$ respectively.
The principle of conservation of energy therefore gives:

$$c_1\rho_1 \cos i . f^2 = c_1\rho_1 \cos i . g^2 + c_2\rho_2 \cos r . h^2 \qquad \text{\_\_\_\_\_(1)}$$

The equation of continuity of displacement at the interface is: $f + g = h$ _____(2)

& from Snell's law: $\frac{\sin^2 r}{\sin^2 i} = \frac{c_2^2}{c_1^2} = \frac{\rho_1}{\rho_2}$ _____(3)

Elliminating h in (1) from (2) and substituting (3) we have:

$$c_1\rho_1 \cos i . f^2 = c_1\rho_1 \cos i . g^2 + c_2\rho_2 \cos r . (f+g)^2$$

$$c_1\rho_1 \cos i . f^2 = c_1\rho_1 \cos i . g^2 + c_2\rho_2 \cos r . f^2 + c_2\rho_2 \cos r . g^2 + c_2\rho_2 \cos r . 2fg$$

$$c_1\rho_1 \cos i . \left(f^2 - g^2\right) = c_2\rho_2 \cos r . \left(f^2 + g^2\right) + c_2\rho_2 \cos r . 2fg$$

$$\div c_2\rho_2 \cos r . 2fg$$

$$\frac{c_1\rho_1 \cos i . \left(f^2 - g^2\right)}{c_2\rho_2 \cos r . 2fg} = \frac{c_2\rho_2 \cos r . \left(f^2 + g^2\right)}{c_2\rho_2 \cos r . 2fg} + 1$$

subst (3)

$$\frac{\sin i}{\sin r}\frac{\sin^2 r}{\sin^2 i}\frac{\cos i}{\cos r}\frac{\left(f^2 - g^2\right)}{2fg} = \frac{c_2\rho_2 \cos r\left(f^2 + g^2\right)}{c_2\rho_2 \cos r . 2fg} + 1$$

$$\frac{\sin r . \cos i}{\sin i . \cos r}\frac{\left(f^2 - g^2\right)}{2fg} = \frac{c_2\rho_2 \cos r\left(f^2 + g^2\right)}{c_2\rho_2 \cos r . 2fg} + 1$$

$\times 2fg$

$$\frac{\sin r.\cos i}{\sin i.\cos r}\left(f^2-g^2\right)=\left(f^2+g^2\right)+2fg$$

factorise $\left(f^2-g^2\right)$ & $\left(f^2+g^2\right)$

$$\frac{\sin r.\cos i}{\sin i.\cos r}(f+g)(f-g)=(f+g)(f+g)-2fg+2fg$$

$$\frac{\tan r}{\tan i}=\frac{\sin r.\cos i}{\sin i.\cos r}=\frac{(f+g)(f+g)}{(f+g)(f-g)}=\frac{(f+g)}{(f-g)}$$

Fresnel equation 1:

$$(f-g)\sin r.\cos i=(f+g)\sin i.\cos r$$

$$f.\sin r.\cos i-g.\sin r.\cos i=f.\sin i.\cos r+g.\sin i.\cos r$$

$$f.\sin r.\cos i-f.\sin i.\cos r=g.\sin i.\cos r+g.\sin r.\cos i$$

$$f(\sin r.\cos i-\sin i.\cos r)=g(\sin i.\cos r+\sin r.\cos i)$$

$$\frac{f}{g}=\frac{(\sin i.\cos r+\sin r.\cos i)}{(\sin r.\cos i-\sin i.\cos r)}=\frac{\sin(i+r)}{\sin(r-i)}$$

$$\frac{g}{f}=\frac{\sin(r-i)}{\sin(r+i)}$$

Fresnel equation 2:

$$(f-g)\tan r=(f+g)\tan i$$

$$f.\tan r-g.\tan r=f.\tan i+g.\tan i$$

$$f.\tan r-f.\tan i=g.\tan i+g.\tan r$$

$$f(\tan r-\tan i)=g(\tan i+\tan r)$$

$$\frac{f}{g}=\frac{(\tan i+\tan r)}{(\tan r-\tan i)}=\frac{\tan(i+r)(1-\tan i.\tan r)}{\tan(r-i)(1+\tan r.\tan i)}=\frac{\tan(i+r)(1-\tan i.\tan r)}{-\tan(i-r)(1-\tan i.\tan r)}=\frac{\tan(r+i)}{\tan(r-i)}$$

$$\frac{g}{f}=\frac{\tan(r-i)}{\tan(r+i)}$$

from Fresnel equation 1

$$\frac{\sin(r-i)}{\sin(r+i)}=\frac{\sin r.\cos i-\cos r.\sin i}{\sin r.\cos i+\cos r.\sin i}$$

Snell's Law:

$$n_1\sin i=n_2\sin r$$

$$\sin r=\frac{n_1}{n_2}\sin i \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(4)$$

subst (4)

$$\frac{\sin(r-i)}{\sin(r+i)}=\frac{\frac{n_1}{n_2}\sin i.\cos i-\sin i.\cos r}{\frac{n_1}{n_2}\sin i.\cos i+\sin i.\cos r}$$

$$=\frac{\sin i\left(\frac{n_1}{n_2}\cos i-\cos r\right)}{\sin i\left(\frac{n_1}{n_2}\cos i+\cos r\right)}$$

$\times n_2$

$$\frac{\sin(r-i)}{\sin(r+i)}=\frac{n_1\cos i-n_2\cos r}{n_1\cos i+n_2\cos r} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(5)$$

subst $\cos r=\sqrt{1-\sin^2 r}=\sqrt{1-\left(\frac{n_1}{n_2}\sin i\right)^2}$ __________________________________(6)

$$\frac{\sin(r-i)}{\sin(r+i)}=\frac{n_1\cos i-n_2\sqrt{1-\left(\frac{n_1}{n_2}\sin^2 i\right)^2}}{n_1\cos i+n_2\sqrt{1-\left(\frac{n_1}{n_2}\sin^2 i\right)^2}}$$

(Fresnel equation for polarization perpendicular to incicent plane: s-polarized) from Fresnel equation 2

$$\frac{\tan(r-i)}{\tan(r+i)}=\frac{\sin(r-i)}{\cos(r-i)}\times\frac{\cos(r+i)}{\sin(r+i)}$$

$$=\frac{(\sin r.\cos i-\cos r.\sin i)(\cos r.\cos i-\sin r\sin i)}{(\cos r\cos i+\sin r\sin i)(\sin r\cos i-\cos r\sin i)}$$

$$=\frac{\sin r.\cos r.\cos^2 i+\sin r.\cos r.\sin^2 i-\cos^2 r\sin i.\cos i-\sin^2 r.\sin i.\cos i}{\sin r.\cos r.\cos^2 i+\sin r.\cos r\sin^2 i+\sin^2 r.\sin i.\cos i+\cos^2 r.\sin i.\cos i}$$

$$=\frac{\sin r.\cos r\left(\cos^2 i+\sin^2 i\right)-\sin i.\cos i\left(\cos^2 r+\sin^2 r\right)}{\sin r.\cos r\left(\cos^2 i+\sin^2 i\right)+\sin i.\cos i\left(\cos^2 r+\sin^2 r\right)}$$

$$=\frac{\sin r.\cos r-\sin i.\cos i}{\sin r.\cos r+\sin i.\cos i}$$

$\div\sin i$

$$=\frac{\frac{\sin r.\cos r}{\sin i}-\cos i}{\frac{\sin r.\cos r}{\sin i}+\cos i}$$

Snell's Law:

$$n_1\sin i=n_2\sin r$$

$$\frac{\sin r}{\sin i}=\frac{n_1}{n_2}\qquad\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(7)}$$

subst (7)

$$=\frac{\frac{n_1}{n_2}\cos r-\cos i}{\frac{n_1}{n_2}\cos r+\cos i}$$

$\times n_2$

$$=\frac{n_1\cos r-n_2\cos i}{n_1\cos r+n_2\cos i}$$

subst (6)

$$\frac{\tan(r-i)}{\tan(r+i)} = \frac{n_1\sqrt{1-\left(\frac{n_1}{n_2}\sin i\right)^2} - n_2\cos i}{n_1\sqrt{1-\left(\frac{n_1}{n_2}\sin i\right)^2} + n_2\cos i}$$

(Fresnel equation for polarization parallel to incident plane: p-polarized)